Collective modes in an open microwave billiard

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Numerical calculations for a microwave Sinai billiard coupled strongly to a lead are performed as a function of the coupling strength between billiard and lead. They prove the formation of different time scales in an open quantum system at large coupling strength. The short-lived collective states are formed together with many long-lived trapped states.

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I. INTRODUCTION

For more than 10 years the redistribution taking place in open quantum systems at strong coupling to the continuum has been studied. It is observed in calculations for nuclei [1], atoms [2], molecules [3], and is discussed from a more general point of view [4]. A common result of all these theoretical studies is that the redistribution in the system consists in the formation of different time scales, i.e., a few short-lived states are formed together with long-lived ones. In [5], it was found that the process of reorganization taking place in an open quantum system has very much in common with the process of self-organization known from other fields of physics. In [6], the problem is studied analytically and numerically for M states coupled to one open decay channel in the limit $M\rightarrow\infty$ and for finite M ($M\geqslant 100$). It could be shown that the redistribution fulfills, in many well-defined cases, the conditions of a second-order phase transition. In such cases, the poles of the S matrix approach an accumulation point with $M\rightarrow\infty$ [7].

The studies are performed theoretically mostly as a function of the coupling strength between system and environment. A clear experimental confirmation for the formation of collective short-lived states at strong coupling in realistic systems is still missing. For that reason we study, in the present paper, the properties of a microwave billiard coupled to a lead as a function of the coupling strength. We find that collective modes are formed, indeed, at a certain critical value of the coupling strength.

In Sec. II, we provide the values which are characteristic for the formation of different time scales in open quantum systems. In Sec. III, the numerical method for calculating the poles of the S matrix and the wave functions of the resonance states is sketched, while the numerical results obtained are represented and discussed in Sec. IV. Some conclusions are drawn in the final section.

II. CHARACTERISTIC QUANTITIES OF THE REDISTRIBUTION IN OPEN QUANTUM SYSTEMS

The behavior of an open quantum system in dependence on the coupling strength of the states to decay channels can be illustrated best by introducing the coupling parameter α which characterizes the average degree of overlapping of the states. Schematically, the Hamiltonian of the system can be written as

$$\mathcal{H}=\mathcal{H}^0-i\alpha VV^\dagger. \tag{1}$$

It is non-Hermitian, while the two parts $\mathcal{H}^0$ and $VV^\dagger$ are Hermitian. The first term is the Hamiltonian of the closed system (corresponding to $\alpha=0$) and V is the vector of the coupling between the discrete eigenstates of $\mathcal{H}^0$ and the decay channels. The eigenvalues $E_i-i/2\ \Gamma_i$ of $\mathcal{H}$ provide the positions E_i and widths Γ_i of the resonance states. The Hamiltonian is symmetric and its eigenfunctions are therefore biorthogonal with the relation $\langle\Phi_i^{\rm left}|\Phi_j^{\rm right}\rangle=\langle\Phi_i^*|\Phi_j\rangle=\delta_{i,j}$ (here and in the following, the $\Phi_i^{\rm right}$ are denoted by Φ_i). A measure for the degree of biorthogonality of the eigenfunctions is given by the standard scalar product $\langle\Phi_i|\Phi_j\rangle$. Generally, under the biorthogonality normalization $\langle\Phi_i^*|\Phi_j\rangle=\delta_{i,j}$ we get $\langle\Phi_i|\Phi_i\rangle\geqslant 1$ and the total deviation from orthogonality is characterized by

$$B\equiv\frac{1}{M}\sum_{i=1}^{M}\langle\Phi_i|\Phi_i\rangle\geqslant 1, \tag{2}$$

where M is the number of states.

A redistribution of the system takes place at a certain critical value of the parameter α since the leading part of $\mathcal{H}$ is $\mathcal{H}^0$ at $\alpha<\alpha_{\rm crit}$ but $VV^\dagger$ at $\alpha>\alpha_{\rm crit}$. The rank of the two parts is different: $\mathrm{rank}(\mathcal{H}^0)=N$, where N is the number of states, while $\mathrm{rank}(VV^\dagger)=K$, where K is the number of open decay channels. Most interesting is the case $K<N$ which is realized in many physical systems. In this case, the redistribution consists in the formation of different time scales. At $\alpha>\alpha_{\rm crit}$, K states align with the K open decay channels and are short-lived while $N-K$ states become orthogonal to these states and are long-lived. The redistribution takes place more or less suddenly at $\alpha\approx\alpha_{\rm crit}$ [1–10]. Since the K aligned states are short-lived, the evolution of the system is, at $\alpha>\alpha_{\rm crit}$, determined by only a small number of states.

The appearance of short-lived collective states is a pure quantum effect caused by the interference of overlapping (in-

dividual) resonance states and occurs in regular as well as in chaotic systems (see, e.g., [10]). It is described by the nondiagonal matrix elements of $\mathcal{H}$ and can be seen most clearly in the case with one open decay channel ($K=1$). In this case, no additional interferences between different channels appear. A semiclassical analysis, however, cannot be performed when the number of channels is very small.

In [6], analytical and numerical studies for $M\to\infty$ as well as for finite M are performed for various systems whose levels are coupled to one decay channel ($K=1$). The main result is the mathematically exact formulation of the conditions under which the redistribution taking place in the system can be considered as a phase transition. The phase transition occurs at a well-defined critical value $\alpha_{\rm crit}$, where the widths bifurcate: the width Γ_{i_0} of one of the states ($i=i_0$, lying in the middle of the spectrum) increases with further increasing α while the widths Γ_i, $i\neq i_0$, of all the other states start to decrease. At $\alpha_{\rm crit}$, the width Γ_{i_0} is much smaller than the length of the spectrum considered.

In the case of an ideal picket-fence model with equal coupling of all the states to the decay channel, the bifurcation of the widths is accompanied by the following changes occurring in the system at $\alpha_{\rm crit}$ [6].

(i) The value Γ_{i_0}/M increases linearly as a function of α with the slope $1/M$ for $\alpha<\alpha_{\rm crit}$ and with the slope 1 for $\alpha>\alpha_{\rm crit}$.

(ii) The value B, which characterizes the biorthogonality of the eigenfunctions, shows a significant maximum at $\alpha=\alpha_{\rm crit}$.

(iii) The number of principal components $N^p_{i_0}$ of the state $i=i_0$ grows suddenly from its minimum value $1/M$ (corresponding to an unmixed state) to the maximum value 1 corresponding to a mixing with *all* states of the spectrum, including those which are lying at a distance much larger than the width Γ_{i_0}.

Here,

$$N^p_{i_0}=\frac{1}{M\sum_{j=1}^{M}|b_{i_0 j}|^4}, \tag{3}$$

where $b_{i_0 j}$ denotes the coefficients describing the decomposition of the vector $|\Phi_{i_0}\rangle$ into the basis $|\Phi^0_i\rangle$ of the eigenvectors of the Hamiltonian $\mathcal{H}^0$. [Note, in Eq. (3) the coefficients $b_{i_0 j}$ are normalized like a probability: $\Sigma_{j=1}^{M}|b_{i_0 j}|^2=1$.] The above statements are not only true for the ideal picket-fence model but also for many other systems under well-defined conditions [6]. The maximum mixing of the state i_0 with all the other states expresses the large collectivity of this state which aligns with the channel. Its width increases with α for $\alpha>\alpha_{\rm crit}$ M times quicker than for $\alpha<\alpha_{\rm crit}$, while the widths of the remaining states decrease with increasing $\alpha>\alpha_{\rm crit}$. This decrease is due to the lack of those components in their wave functions which are aligned with the channel and transferred to the state $i=i_0$. In other words, their wave functions become orthogonal to the channel wave functions, whereas the fast decaying state aligns with it [8]. The $M-1$ states $i\neq i_0$ are called *trapped* states.

On the basis of these results, it was possible to identify the changes of the system occurring at $\alpha_{\rm crit}$ with a second-order phase transition [6]: the first derivative of Γ_{i_0}/M jumps at the critical value $\alpha_{\rm crit}$ and the (normalized) partial width $(\Gamma_{i_0}/M)\langle\Phi_{i_0}|\Phi_{i_0}\rangle$ fluctuates and may be considered as the order parameter. The rearrangement of the system at $\alpha_{\rm crit}$ is a *collective* effect to which all states of the system contribute in spite of the fact that the widths of all of them are much smaller than the length of the spectrum. The maximum of B at $\alpha_{\rm crit}$ is an expression for the avoided crossings of many levels which appear *simultaneously* near $\alpha_{\rm crit}$. In other words, there is a multiple level repulsion in the complex energy plane which occurs, in the limit $M\to\infty$, exactly at $\alpha_{\rm crit}$. In the schematic model used in [6], the states decouple for large α, i.e., B decreases for $\alpha>\alpha_{\rm crit}$ and approaches the value 1.

A reorganization in an open quantum system may take place also in those cases in which the conditions for a phase transition are not fulfilled [6,8]. In these cases, the reorganization process takes place locally, the Γ_{i_0}/M and $N^p_{i_0}$ increase smoothly as a function of α also for large M, and $B\approx 1$ for all α. A collective state is formed also in these cases but its wave function contains components only from those basis states (eigenfunctions of $\mathcal{H}^0$) which are lying in the energy region of its width. That means it interacts only with those states which are overlapped by it in an energy region according to its decay width. The level repulsions in the complex plane occur successively with increasing α and increasing energy region overlapped by the state i_0. They continue to take place with $M\to\infty$ [6]. These results underline once more the meaning of the values Γ_{i_0}/M, B, and $N^p_{i_0}$ for the characterization of the redistribution taking place in an open quantum system at high level density.

The schematical model with the Hamiltonian (1), of course, cannot be expected to describe all properties of a realistic system with strongly overlapping resonances (large α). The decoupling of the short-lived and long-lived states, expressed by $B\to 1$, at large α occurs only under certain model assumptions, see, e.g., [5,9]. The poles of the S matrix provide, however, the positions E_i and widths Γ_i of the resonance states even if they overlap strongly. For a microwave billiard, the physical meaning of the poles of the S matrix has been demonstrated by means of the Wigner time-delay function [11].

III. CALCULATIONS FOR THE OPEN MICROWAVE BILLIARD

We consider a two-dimensional flat resonator coupled to a waveguide and solve the equation

$$-\Delta\Phi=E\Phi. \tag{4}$$

We use the Dirichlet boundary condition, $\Phi=0$, on the border of the billiard and of the waveguide. The waveguide has a width equal to 1 and is attached to the resonator through an opening with a width w which is described also by the Dirichlet boundary condition. For $w=0$ the resonator and the

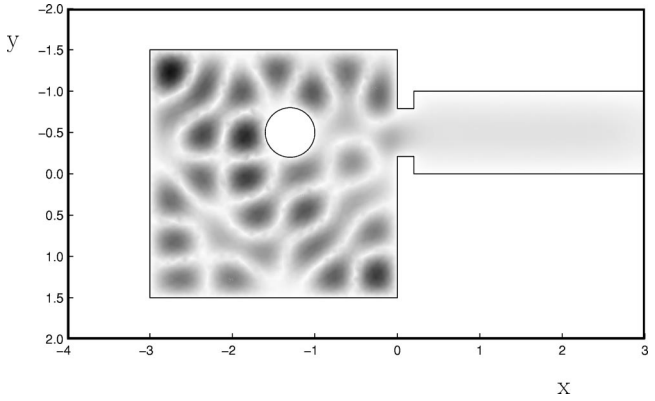

FIG. 1. Wave function $|\Phi_i|^2$ of one of the trapped resonance states for $w=0.58$. The x,y, and w are given in arbitrary units $[x]$.

waveguide are disconnected, while $w=1$ represents the maximal coupling. The shape of the system is shown in Fig. 1.

The wave function inside the waveguide has the asymptotic form

$$\Phi=[e^{ikx}-R(E)e^{-ikx}]u(y). \tag{5}$$

Here $u(y)$ is the transversal mode in the waveguide, k is the wave number, and $R(E)$ is the reflection coefficient.

The energies and widths of the resonance states are given by the poles of the coefficient $R(E)$ analytically continued into the lower complex plane. To find the poles we use the method of exterior complex scaling [12]. The general idea is to study the system after a scaling transformation is applied to the x coordinate, see [12,13]: $x\rightarrow\tilde{x}=g(x)$. The function g is chosen as

$$g(x)=\begin{cases} x, & x\leqslant x_0 \\ \theta f(x), & x>x_0 \end{cases} \tag{6}$$

with $f(x)$ such that $g(x)$ is three times differentiable and the inverse transformation $g^{-1}(\tilde{x})$ exists. The attached waveguide extends to ∞ parallel to the x axis and we choose x_0 to be localized inside it. The related transformation of the wave function reads

$$\Phi(x,y)\rightarrow\frac{1}{\sqrt{g'(\tilde{x})}}\tilde{\Phi}(\tilde{x},y). \tag{7}$$

Using it, Eq. (4) becomes

$$\left[-\frac{\partial}{\partial\tilde{x}}\left(\frac{1}{g'^2}\frac{\partial}{\partial\tilde{x}}\right)-\frac{\partial^2}{\partial y^2}\right]\tilde{\Phi}(\tilde{x},y)+\left(\frac{2g'g'''-5g''^2}{4g'^4}\right)\tilde{\Phi}(\tilde{x},y)$$
$$=E\tilde{\Phi}(\tilde{x},y). \tag{8}$$

For a real parameter θ, this equation is fully equivalent to Eq. (4) since the transformation (7) is unitary. Moreover, the two equations are fully identical for $x<x_0$. Since x_0 lies inside the waveguide, the shape of the resonator is not changed by the transformation (7) which only rescales a part of the x axis related to the waveguide. Moreover, since the waveguide is oriented parallel to the x axis, the transformation does not change the boundary of the system. For θ complex, Eq. (7) ceases to be unitary and the spectral properties of Eqs. (4) and (8) are different. The continuous spectrum of Eq. (4) is localized at $\langle\pi^2,\infty)$, whereas the continuous spectrum of Eq. (8) is rotated into the complex plane and is equal to

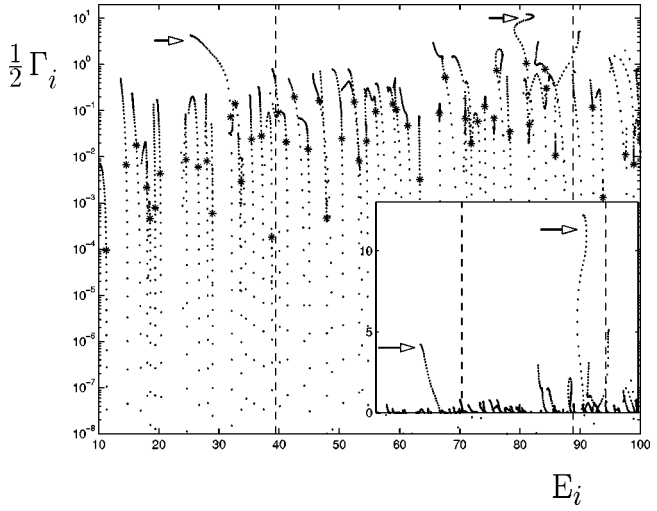

FIG. 2. Eigenvalue picture: motion of the poles of the S matrix as a function of w. The positions of the poles at $w_{\rm crit}$ are marked by stars. The positions of the two thresholds are shown by dashed lines. The energies and widths are given in units of $[x]^{-2}$. Note the logarithmic scale of the ordinate. In the inset, the scale of the ordinate is linear.

$$\cup_{n=1:\infty}\{(n\pi)^2+\theta^{-2}\langle 0,\infty)\}. \tag{9}$$

This is a union of half-lines representing the continuous spectrum starting out from the real axis at every threshold energy $(n\pi)^2$ with an angle $-2\arg\theta$. (We have used $\theta=1+i$ in the numerical calculations.) The rotated continuous spectrum uncovers additive complex eigenvalues of Eq. (8), the positions of which are independent of θ. These eigenvalues coincide exactly with the poles of the analytically continued reflection coefficient $R(E)$ [12,13]. The corresponding eigenfunctions are denoted usually as resonance states. The (normalized) wave function $|\Phi_i|^2$ of one of the trapped resonance states is plotted in Fig. 1. In this case the waveguide was coupled to the resonator through an opening with width $w=0.58$.

IV. COLLECTIVE MODES IN THE MICROWAVE BILLIARD

We discuss the results obtained for the open billiard shown in the preceding section. Most important is the formation of collective states which appear suddenly as a function of the opening size w of the billiard.

Figure 2 shows the resonances and the interplay between the different states with increasing w. As long as w is small, all Γ_i increase slowly as a function of w. At the value $w_{\rm crit}=0.44$, the widths of two states marked by arrows ($i=1,2$) start to separate from the widths of the other states. The first broad state $i=1$ arises between the first and second threshold, while the second one emerges between the second and

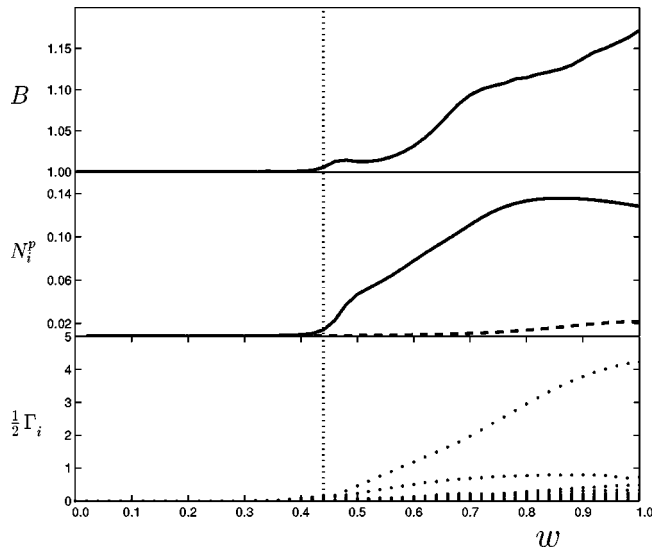

FIG. 3. B, N_i^p (for $i=1$ [full line] and one of the trapped states [dashed line]) and $\Gamma_i/2$ (for all states i between the first and second threshold) as a function of w. The critical value w_{crit} is indicated by the dotted line. The total number of states M taken into account is equal to 120. The widths are given in units of $[x]^{-2}$ and those of w in $[x]$.

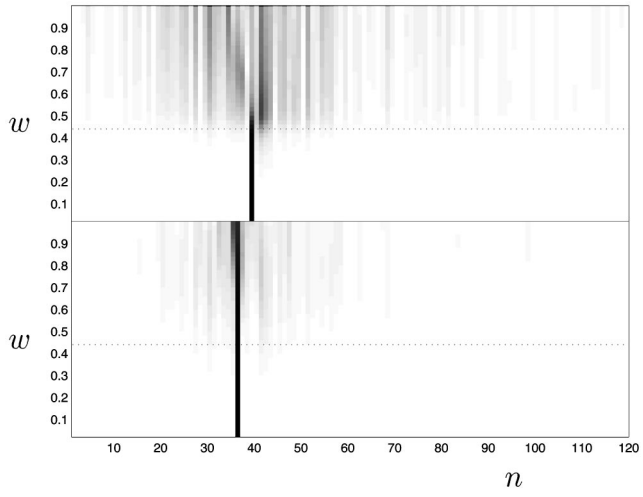

FIG. 4. The spreading of the coefficients of the wave functions of the first broad state Φ_1 (above) and of a typical trapped state (below) over the first 120 bound states of the closed resonator. The critical value w_{crit} is marked by the horizontal dotted lines. The units of w are given in $[x]$.

third threshold. Both appear at almost the same critical value $w_{\mathrm{crit}} \approx 0.44$ (marked by a star at the trajectories of the poles). This is due to the fact that the reorganization of the system takes place at a certain critical value of $\overline{\Gamma}/(K\overline{D})$ if the states are coupled to all K channels with, on the avarage, the same strength [3,6]. Here $\overline{\Gamma}$ is the mean width, K the number of open channels, and $\overline{D}$ the average distance between the resonance states. In our case, $\overline{D}$ does not depend on K and $\overline{\Gamma} \propto K$, see Fig. 2. Thus, the value $\overline{\Gamma}/(K\overline{D})$ is independent of K and $w_{\mathrm{crit}}^{i=1} \approx w_{\mathrm{crit}}^{i=2}$. Note that the widths of *all* states are much smaller than the length of the spectrum for all values of w (see Fig. 2).

More information on the mechanism of the formation of the two broad states can be obtained from the $B(w)$, Eq. (2), and $N_{i=1,2}^p(w)$, Eq. (3). In Fig. 3 we see a sudden increase of $N_1^p(w)$ at the critical value w_{crit} where the width Γ_1 starts to separate. For comparison, $N_{i\neq 1}^p(w)$ for a typical trapped state which does not increase at w_{crit} is plotted by a dashed line. In accordance with the increase of N_1^p at w_{crit}, the coefficients $|b_{1,i}|^2$ start to spread at w_{crit}. This is illustrated in Fig. 4, where the spreading of the coefficients of $\Phi_{i=1}$ as a function of w is shown. For comparison, the spreading of the coefficients for the above-mentioned trapped state is also shown in Fig. 4. It starts to increase only at $w \gg w_{\mathrm{crit}}$. Analogous results are obtained for the second broad state $i=2$.

The results represented in Figs. 2 and 3 can be understood in the following manner. At w_{crit}, there is first an interference of neighbored states which are coupled with a relatively large strength to the corresponding channel. Then, interferences of these states with weaker coupled ones occur. All the interfering states are spread over an energy region being much larger than each of their widths. As a result, two of the states, $i=1$ and 2, collect in their wave functions a great deal of components which are aligned with *either* of the two channels. The widths of these two states increase strongly with increasing $w > w_{\mathrm{crit}}$ while the widths of the remaining $M-2$ states increase much slower or decrease (Fig. 2). The position of the thresholds plays a subordinate role for the mixing of the resonance states (Fig. 4).

In Fig. 3, also the biorthogonality measure B is shown. It starts to increase near w_{crit}. This is a hint of the interferences at $w \approx w_{\mathrm{crit}}$ between several levels. The $B(w)$ increase further with $w > w_{\mathrm{crit}}$ because of the mutual interference of all states in this energy region. Thus, the B reflect the interplay between the states shown in Fig. 2. (It should be noted that the biorthogonality $\langle \Phi_i^* | \Phi_j \rangle = \delta_{i,j}$ is fulfilled in our calculations to a high accuracy for all w.)

As to the interpretation of the results from the point of view of a phase transition, one can state the following: the collective state $i=1$ (corresponding to the one-channel case) contains contributions in its wave function from states lying at a distance *much larger* than covered by its width. The mixing occurs in the relatively small interval of w between 0.44 and 0.6. Nevertheless, the number of interfering states, which contribute to the collectivity of the state $i=1$, is limited and mutual interferences of all states take place at $w > 0.7$.

Similar results are obtained for a regular billiard, i.e., for a rectangular billiard without the circle. In this case, many states are degenerate at $w=0$. The degeneracy is removed by the coupling to the lead and collective modes are formed in the same manner as in the case considered above. This result underlines once more the decisive role the interferences between *individual* resonance states play for the formation of the collective states.

V. CONCLUSION

Summarizing the results, we conclude that short-lived collective modes can arise in an open microwave billiard coupled to a lead. In the case considered, they appear suddenly at the critical value w_{crit} of the parameter w by which

the degree of opening of the billiard is characterized. The mechanism of their formation is understood on the basis of the interference of resonance states which are spread over a large energy region. The collective states contain contributions in their wave functions from many states including those lying at energies much more distant than the range covered by their widths. They appear together with long-lived trapped states. The appearance of the different time scales should be visible in the time-delay function studied as a function of w. Further investigations of the phenomenon are in progress.

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